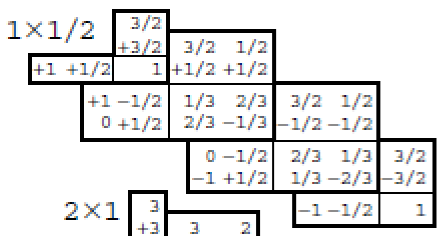
Clebsch-Gordan Coefficient Example
Phys 402

The eigenfunctions of J^2 can be expressed as linear combinations of states with different values of m_ℓ and m_s using the world-famous Clebsch-Gordan coefficients ($C^{\ell \;\; s \;\; j}_{m_\ell \; m_s \; m_j}$) as:

$$\left| j\ m_j \right\rangle = \sum_{m_\ell + m_s = m_j} C^{\ell \;\; s \;\; j}_{m_\ell \; m_s \; m_j} \left| \ell\ m_\ell \right\rangle \left| s\ m_s \right\rangle \qquad [4.185]$$

Where the ket $\left| \ell\ m_\ell \right\rangle$ represents the spherical harmonics $Y_\ell^{m_\ell}$. The C-G coefficient values are given in Table 4.8 on page 179 of Griffiths. Remember that all of the coefficients should appear under a square root, with the minus sign (if any) out front.

Now for an example of how to construct states that are simultaneous eigenfunctions of L^2, S^2, J^2 and J_z. Take the case again of hydrogen with $\ell = 1$ and spin $s = 1/2$. How do we find the state with $j = 3/2$ and $m_j = -1/2$ in terms of the $Y_\ell^{m_\ell}$ and spinors? Look at the $1 \times 1/2$ CG Table on page 179. We are led to this table because we are combining an angular momentum vector with $\ell = 1$ and spin vector with $s = 1/2$.



Now look under the column labeled " $\begin{matrix}3/2\\-1/2\end{matrix}$ ". It says:

$$\left| \frac{3}{2}\ -\frac{1}{2} \right\rangle = \sum_{m_\ell + m_s = -1/2} C^{1 \;\; 1/2 \;\; 3/2}_{m_\ell \; m_s \; -1/2} \left| 1\ m_\ell \right\rangle \left| \frac{1}{2}\ m_s \right\rangle$$

$$\left| \frac{3}{2}\ -\frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} \left| 1\ 0 \right\rangle \left| \frac{1}{2}\ -\frac{1}{2} \right\rangle + \sqrt{\frac{1}{3}} \left| 1\ -1 \right\rangle \left| \frac{1}{2}\ \frac{1}{2} \right\rangle$$

This can be written in a more familiar way in terms of spherical harmonics and spinors as:

$$\langle \vec{r} | \frac{3}{2}\ -\frac{1}{2} \rangle = \sqrt{\frac{2}{3}}\, Y_1^0(\theta,\phi)\chi_- + \sqrt{\frac{1}{3}}\, Y_1^{-1}(\theta,\phi)\chi_+$$

One can move back and forth between the coupled and un-coupled representations using the Clebsch-Gordan table on page 179. Here is the schematic layout for the CG table for combining two spins (called $\vec{S}_1$, $\vec{S}_2$) to form a total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ (S^2 has eigenvalue $s(s+1)\hbar^2$):

General Schematic of the C-G Table

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$S_1 \times S_2$

s

m_s

Coupled Representation

$m_{s_1} \quad m_{s_2}$

$CG\#$

Un-Coupled Representation